**Eigenvalues and Eigenvectors**

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The topic of eigenvalues and eigenvectors is actually rather simple. However, understanding it properly requires a solid understanding of linear transformations, determinants, linear systems and basis changes.

Consider a linear transformation which transforms the basis vector to and to . Any vector created on the original basis would most likely get knocked off its span after the transformation. However, some vectors do remain on their original span, meaning the transformation only manages to stretch or squish the vector. In this example, itself is one such vector, that is stretched to three times its original length, but remains in the same span. Due to the way linear transformation works, any scalar multiple of this vector also behaves in the same way. The vector is another vector that also remains on its span, getting stretched by a factor of . Again, any vector on the diagonal line spanned by this vector also remains in the span.

For this example, those two vectors (and their scalar multiples) are the only ones that remain on their span. Any other vector will get knocked off its span. These special vectors are called eigenvectors. Any eigenvector also has a value associated with it called an eigenvalue, which is the factor by which the vector is stretched or squished. Eigenvalues can also be negative, which mean the vector was flipped.

To see why eigenvalues and eigenvectors might be useful, think about a 3D rotation. If we can find an eigenvector for that rotation, what we have found is the axis for the rotation. It is, of course, far easier to think of a 3D rotation as a rotation around some vector by some degree, rather than in terms of the complete transformation matrix associated with it. In this case, the eigenvalue would be , since nothing is stretched or squished.

## The Eigenvalue Equation

The main idea of eigenvectors is captured by the equation , where is the transformation matrix, is the eigenvector and is the eigenvalue. Thus, it all comes down to being able to find the values of and that make this equation true.

The equation is a little awkward to work with, because the left-hand side is matrix multiplication while the right-hand side is scalar multiplication. To make it easier, we can write as a matrix. This is simply , or the identity matrix with s down the diagonal instead of s. Thus, we can re-write the equation like this:

This is the main equation for this topic, the eigenvalue equation or the characteristic equation. This new matrix, , will look something like . We are trying to find the vector which, when multiplied by this matrix, gives the vector. For this to be true, the determinant of this matrix has to be . We can now tweak the value of until we find the value for which this is true. In the example we are using, this value is or . Plugging these into the matrix, we can easily find the vector for which we get a vector by calculating the null space. itself is of course an answer to this, but that would make all of our work meaningless so we ignore that answer for now.

There are a few things to keep in mind about eigenvalues and eigenvectors.

* An eigenvectors (under normal circumstances)
* The sum of the eigenvalues for a matrix will always be equal to the sum of the diagonal values of the matrix.
* The product of the eigenvalues of a matrix will always be equal to the determinant of the matrix.

Consider a symmetric matrix . For symmetric matrices, the eigenvectors will come out to be perpendicular to each other, and the eigenvalues will be nice, real numbers. In fact, if we begin to solve this, the quadratic equation we get will be where is called the ‘trace’ of the matrix , which is simply the sum of the diagonals, and is the determinant of . Everything about this solution is extremely convenient. The eigenvalues are and and the corresponding eigenvectors come out to be and .

Consider a permutation matrix . The eigenvalues here would be and and the eigenvectors would be and respectively. Notice that the matrix in the previous example, is simply . Notice that the eigenvalues also had added to them, and that the eigenvectors remained unchanged. Thus, for any matrix, if we add to it, the eigenvalues increase by and the eigenvectors remain unchanged. This can be proven in a straightforward manner using the eigenvalue equation.

However, consider an example where we add some other matrix which has the eigenvalue . We cannot say that by adding to we are adding α to , which is . This is because we have no reason to believe that is also an eigenvector for . The same is true for the multiplication with . In these situations, we need to find the eigenvalues and eigenvectors manually.

Consider a projection matrix that projects the vector onto a plane, giving a vector . In that situation, (and its scalar multiples) would all be eigenvectors, since they remain unchanged. Another eigenvector would be the vector that is perpendicular to the plane. Thus, we have two things to look at here, , where the eigenvalue is , and , where the eigenvalue is .

There are situations where an eigenvalue is is singular.

There are situations where there are no eigenvectors or eigenvalues. In those cases, the values of will be imaginary. Imaginary eigenvalues indicate some form of rotation, such as a rotation by , which will indeed give us the eigenvalues and . In these situations, of course, the eigenvalues are complex conjugates just like any other imaginary number would have.

There are situations where we have a single eigenvalue for a line full of eigenvectors. Consider a shear matrix and there is only one eigenvector, due to the repeated eigenvalues we get from the eigenvalue equation. Situations like these cause problems, which we will be looking at in the next lecture.

There are also situations where there is a single eigenvalue, but the eigenvectors do not lie on the same line. For example, the matrix stretches everything by , and has an eigenvalue of. Every single vector in the plane is an eigenvector in this case.

In general, when we are given a symmetric matrix, the results we will get with eigenvalues and eigenvectors will all be extremely convenient to work with. As we move further away from a symmetric matrix, situations will begin to get increasingly complex. Anti-symmetric matrices, such as a rotation matrix, are the worst, since eigenvalues and eigenvectors do not ‘exist’ at all, i.e. they are imaginary.Diagonalization

Suppose we have independent eigenvectors,, , , , for a certain matrix . If we take each of these eigenvectors to be a column of a matrix, , we have a matrix that we can call the eigenvector matrix.

Now consider what happens when we multiply .

Since we took independent eigenvectors, we know that has an inverse. Thus, we can write

where (lambda) is the diagonal eigenvalue matrix, which has the eigenvalues down its diagonal.

This process is called diagonalization. We could also write the equation for diagonalization the other way around, as

Consider the matrix . Notice that the eigenvalues for are just the square of the eigenvalues for , and that the eigenvectors remain the same. We can show this using the formula for diagonalization as well. Since . This says the same thing, but in matrix form. Since is unchanged, the eigenvectors must be the same and since is squared, the eigenvalues are squared. Of course, this applies for as well. Thus, the diagonalization process can make squaring matrices very easy, whereas doing it normally would be a nightmare.

If , then we will find and . This gives us the eigenvectors and . Thus,

However, not all matrices will have independent eigenvectors. If all the are different, has independent eigenvectors and is therefore diagonalizable. If there are repeated s, the matrix may or may not have independent eigenvectors.

Notice that this is not a completely negative sentence. For example, the identity matrix is a matrix that has repeated s, all of which are . However, there is no shortage of independent eigenvectors for the identity matrix. Every single vector is an eigenvector for the identity matrix. In fact, if is the identity matrix, notice that just gives us back the identity matrix, since that matrix was already diagonalized. A diagonal matrix already has the eigenvalues down the diagonal.

A triangular matrix also has its eigenvalues down the diagonal, but in this case, we will face problems if we have repeated eigenvalues. Say . Of course, the values of are and here. We can check this by finding . We say that the algebraic multiplicity is . However, if we try to find the eigenvectors, we see that which gives us just one eigenvector. We say that the geometric multiplicity is . In situations like this, the diagonalization formula we saw will not work.

Let us go back to a normal situation, where we have a diagonalizable matrix. Say we are given a vector and each increment of is multiplied by a matrix , i.e. . We want to find the th increment, . Here, notice that and so on. Thus, . We can find , and use that to find .

We will see an example of the usage of what we just saw, with the help of the Fibonacci numbers. The Fibonacci numbers are a set which starts with , and continues with each number being the sum of the previous two. Thus, the set begins , , , , , , , , . Now, we want to find .

We know that . This is an equation and not a linear system, so we need to turn it into one. Let there be a second equation simply so that we can work with a system. Using the artificial system of two equations we have just created, we can say

Notice how this satisfies both the equations, and is therefore valid.

For the matrix ,

Thus, . The numerical values are approximately and . This tells us that is definitely diagonalizable, since the values of are different. From these, we can find the two eigenvectors.

This gives us .

This gives us .

Thus, , found using the co-factor method and .

Since , starting at and repeatedly applying this gives us

We know that . Thus,

Thus, .